

# Exact Results for N=2 Compactifications of Heterotic Strings

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We search for  $N = 2$ ,  $d = 4$  theories which can be realized both as heterotic string compactifications on  $K_3 \times T^2$  and as type II string compactifications on Calabi-Yau threefolds. In such cases, the exact non-perturbative superpotential of one string theory is given in terms of tree level computations in the other string theory. In particular we find concrete examples which provide the stringy realization of the results of Seiberg and Witten on N=2 Yang-Mills theory, corrected by gravitational/stringy effects. We also discuss some examples which shed light on how the moduli spaces of different N=2 heterotic vacua are connected.

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## 1. Introduction

Recently, there has been tremendous progress in our understanding of  $N=2$  supersymmetric field theories [1,2,3,4,5,6] and  $N=2$  supersymmetric type II string compactifications [7][8][9]. At the same time, our knowledge of strong-coupling phenomena in string theory has been enriched by exciting work on S-duality and six-dimensional string-string duality [10,11,12,13,14,15,16].

In this paper, we explore  $N=2$  supersymmetric vacua of the heterotic string in four-dimensions. We find examples which suggest that many naively distinct heterotic  $N=2$  moduli spaces are in fact connected, in a way which is very analogous to the way the different type II Calabi-Yau compactifications are connected [8][17]. Moreover as we will argue some  $N = 2$  vacua have dual realizations as both type II and heterotic compactifications. This has dramatic implications, as has been suggested by Strominger.

One of the key ideas leading to the resolution of the conifold singularity is the postulate by Strominger that the absence of neutral perturbative couplings between vector multiplets and hypermultiplets survives nonperturbative string effects. Since the dilaton is part of a hypermultiplet in type II compactifications on Calabi-Yau threefolds, it follows that the tree level prepotential for vector multiplets is exact<sup>1</sup>, while that of the hypermultiplets might get corrected. In the context of  $N = 2$  compactifications of heterotic strings, since the dilaton now sits in a vector multiplet, it follows from Strominger's postulate that the moduli space of hypermultiplets is exact at the tree level while the vector multiplets can receive quantum corrections. This also implies that if we find an  $N = 2$  string compactification which has realizations both as a type II and as a heterotic string compactification, then the exact prepotential for the vector multiplets can be computed using the type II realization at tree level, and the exact hypermultiplet superpotential can be computed using the heterotic realization at tree level. In particular for our examples with dual realizations we compute the non-perturbative corrections to the prepotential for the vector multiplets on the heterotic side, using the known prepotentials of Calabi-Yau threefolds (thus realizing the speculation by many physicists that there should be a connection between quantum moduli spaces of heterotic strings and special geometry of Calabi-Yau threefolds [18][3][19]). The examples we consider include models which have

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<sup>1</sup> In general, different superpotential terms receive contributions at only a specific order in the genus expansion, depending on their modular weight. The moduli space geometry is fixed at genus 0.

$SU(2)$  as well as  $SU(3)$  enhanced gauge symmetry points at the heterotic string tree level. This in particular gives us a string realization, including gravitational corrections, of [1].

It would be desirable to have more examples of this  $N = 2$  heterotic/type II duality. In particular the string-string duality in 6 dimensions relating type IIA compactifications on  $K_3$  to toroidal compactifications of heterotic strings may provide a hint of how dual pairs can be constructed more systematically [13], which seems to lead to other 4 dimensional  $N = 2$  type II/heterotic dual examples [20].

In §2 we briefly review the construction of general (0,4) heterotic string compactifications on  $K_3 \times T^2$  (in [21] similar techniques were used to construct anomaly free six dimensional chiral gauge theories). We then discuss several examples in §3, and point out that “stringy” enhanced gauge symmetry points may provide a way of connecting many or all  $N=2$  supersymmetric heterotic vacua (including those on asymmetric orbifolds [22]). In §4 we come to the main focus of this paper. We construct heterotic theories for which we can propose a specific candidate dual Type IIA string compactification on a Calabi-Yau manifold, and in some cases we give strong evidence for such a duality. We give our conclusions in §5.

The importance and implications of duality between  $N = 2$  compactifications of type II and heterotic strings, as well as the dual interpretation of the conifold singularities as Seiberg-Witten monopole points, has been independently noted by Ferrara, Harvey and Strominger.

## 2. Models on $K_3 \times T^2$

There are several different approaches one might take to constructing  $N=2$  heterotic string vacua. The most straightforward is perhaps to compactify the  $E_8 \times E_8$  or  $SO(32)$  heterotic string on  $K_3 \times T^2$ . We will first discuss this class of compactifications, and then in §3 will discuss modifications of such vacua based on asymmetric orbifolds [22].

The most familiar way of compactifying the heterotic string on  $K_3 \times T^2$  is simply to use the “standard embedding,” equating the spin connection of the manifold with the gauge connection. For the  $E_8 \times E_8$  string this yields a theory (at generic points in the moduli space of the torus) with  $E_7 \times E_8 \times U(1)^4$  gauge group, 10 hypermultiplets in the **56** of  $E_7$ , and 65 gauge neutral moduli hypermultiplets – 20 moduli of the  $K_3$  surface and 45 moduli of the gauge bundle.

Of course, the adjoint scalar fields sitting in the vectors corresponding to the Cartan subalgebra of the gauge group are also moduli which can be given nonzero VEVs. At *generic* points in the moduli space of these scalars, the theory is characterized by 65 hypermultiplets and 19 vectors (18 coming from vector multiplets including the multiplet of the dilaton, one coming from the graviphoton). In the following we will denote this by  $(65, 19)$  and in general will describe the spectrum of a theory which at generic points has  $M$  hypermultiplets and  $N$  vector fields (including graviphoton) by  $(M, N)$ . We note now that in the context of Type IIA strings, such a theory would arise from compactification on a Calabi-Yau manifold with hodge numbers  $b_{11} = N - 1, b_{21} = M - 1$ . The additional hypermultiplet and vector necessary to obtain agreement with the heterotic spectrum would then come from the dilaton and the graviphoton, respectively.

It is useful to review at this point how one could derive this spectrum just using index theory. Let us start with an unbroken gauge group  $\mathcal{G}$  in ten dimensions and break it to a subgroup  $G$  by giving gauge fields on  $K_3$  an expectation value in  $H$  where  $G \times H \subset \mathcal{G}$  is a maximal subgroup.

Compactify further on a torus to get a four dimensional  $N=2$  theory. That part of the matter spectrum which arises from the higher dimensional gauge multiplet can be determined as follows. Decompose

$$adj \mathcal{G} = \sum_i (M_i, R_i) \quad (2.1)$$

where  $M_i$  and  $R_i$  are representations of  $G$  and  $H$  respectively. Then it follows from the index theorem that generically the number of left-handed spinor multiplets transforming in the  $M_i$  representation of  $G$  is given by

$$N_{M_i} = \int_{K_3} -\frac{1}{2} \text{tr}_{R_i} F^2 + \frac{1}{48} \text{dim} R_i \text{tr} R^2 = \text{dim} R_i - \frac{1}{2} \int_{K_3} c_2(V) \text{index}(R_i). \quad (2.2)$$

Here  $V$  denotes the  $H$  bundle parametrizing the VEV of the vacuum gauge fields on  $K_3$ . In addition, for compactifications on  $K_3$  there is a universal contribution to the spectrum of matter hypermultiplets coming from the higher dimensional gravitational fields; this consists of 20 gauge singlet hypermultiplet moduli.

For the case of the standard embedding, we have chosen  $V$  to be an  $SU(2)$  bundle with  $\int_{K_3} c_2(V) = 24$ . Going through the computations above yields the expected 10 **56**s and 45 extra moduli hypermultiplets arising from the higher dimensional gauge fields. But in general we are free [23][24] to choose more general stable, holomorphic  $SU(N)$  bundles

over  $K_3$  in the process of compactification (this corresponds to  $H = SU(N)$  in the previous discussion), subject only to the constraints

$$c_2(V) = c_2(TK_3), \quad c_1(V) = 0. \quad (2.3)$$

More generally, we may wish to choose several different factors  $V_a$  in the vacuum gauge bundle and e.g. embed them in different  $E_8$  factors. In this case the constraints are simply

$$\sum c_2(V_a) = c_2(TK_3), \quad c_1(V_a) = 0. \quad (2.4)$$

When computing the generic spectrum, one only needs to know the number of gauge neutral moduli hypermultiplets and the rank of the gauge group. The gravitational contribution is a universal 20, while the number of moduli of an  $SU(N)$  bundle with  $\int_{K_3} c_2(V) = A$  is  $AN + 1 - N^2$ . Using these formulas (and the knowledge that embedding an  $SU(N)$  bundle in the gauge group will reduce the rank by  $N - 1$ ) it is easy to find the generic spectra of models of this type.

### 3. Examples and Observations about their Moduli Spaces

Let us discuss some of the different theories we may obtain in this way. Embedding a single  $SU(N)$  factor with  $\int_{K_3} c_2(V) = 24$  in one  $E_8$  of the heterotic string breaks this  $E_8$  to  $E_7$ ,  $E_6$ ,  $SO(10)$ , or  $SU(5)$  for  $N = 2, 3, 4, 5$  and results in theories with the following generic spectra:

$$N = 2 : (65, 19) \quad N = 3 : (84, 18) \quad N = 4 : (101, 17) \quad N = 5 : (116, 16) . \quad (3.1)$$

One can also compute the spectrum of charged fields in these models at the points where nonabelian gauge symmetry is classically restored, using the techniques outlined in §2.

The different models listed in (3.1) are of course not unrelated. Starting with the (65, 19) model, one knows that classically there is a point where an  $E_7$  gauge symmetry is restored and that the spectrum there includes 10 **56**s of  $E_7$ . Under the maximal  $E_6 \times U(1)$  subgroup of  $E_7$ ,  $\mathbf{56} = \mathbf{27} + \overline{\mathbf{27}} + \mathbf{1} + \mathbf{1}$  with the  $E_6$  singlets charged under the  $U(1)$ . Moving to the codimension one locus in the moduli space of the Cartan vectors where the scalar in that  $U(1)$  vector multiplet has vanishing VEV, one also therefore finds an extra 20 massless hypermultiplets, charged under that  $U(1)$ . One can now Higgs the  $U(1)$ , leaving 19 extra gauge singlet fields – in other words, one is now on a branch of moduli space with

spectrum  $(84, 18)$ , the  $N=3$  case of (3.1). One can similarly move from the  $N=3$  case to the  $N=4$  case and so forth.

It is very amusing to note the similarity between going from the  $N=4$  to  $N=5$  case and the process described in [8] moving from the moduli space of a Calabi-Yau with hodge numbers  $b_{11} = 2, b_{21} = 86$  to the moduli space of the quintic with  $b_{11} = 1, b_{21} = 101$ . Note that these are the same numbers as we would have gotten from  $N = 4$  and  $N = 5$  above except by an overall shift of 14 in both  $b_{11}$  and  $b_{21}$ . The  $N=4$  model has a special point in the moduli space of vectors where there is an  $SO(10) \times E_8 \times U(1)^4$  gauge symmetry. The charged spectrum there includes 16 **16**s of  $SO(10)$ . Under the  $SU(5) \times U(1) \subset SO(10)$ , one has  $\mathbf{16} = \mathbf{10} + \bar{\mathbf{5}} + \mathbf{1}$  where the  $\mathbf{1}$  is charged under the  $U(1)$ . So on a locus of codimension one in the moduli space of the vectors of the  $N=4$  model, one finds 16 massless hypermultiplets of fields charged under a single  $U(1)$  gauge symmetry. Higgsing this gauge symmetry leads to 15 more neutral hypermultiplets and one less vector multiplet. This is precisely the “mirror” description of the process described in [8]! The numbers of vector and hypermultiplets are however shifted by 14, making it problematic to conjecture a precise duality between this heterotic process and the type II process described in [8]. However, if one shifts the numbers of vector and hypermultiplets by 14 in each of the  $N=2,3,4,5$  cases listed in (3.1) one notices that all of them yield numbers which would arise in type IIA compactification on complete intersection Calabi-Yaus in products of projective spaces [25]. Since it is precisely this class of manifolds which we know are connected by conifold transitions, it would be interesting to see if one could somehow explain the shift of 14 and find a precise duality between these heterotic theories and some type II examples.

We would like to find  $(0,4)$  heterotic compactifications for which we *can* find a dual type II compactification on a Calabi-Yau threefold, and for which we can give a very stringent test of the duality. While there are known Calabi-Yau manifolds with the requisite hodge numbers to produce the numbers of vector and hypermultiplets of some of the theories in (3.1) when used to compactify type II strings, these examples are too complicated to provide a good testing ground for such a duality conjecture. We will come back to this point, and provide much better examples where we can conjecture and give extremely strong evidence for such a duality. But first, we find it worthwhile to discuss the connectedness of the moduli space of  $(0,4)$  heterotic theories.

It is now strongly believed that type II compactifications on different Calabi-Yau manifolds are connected smoothly [7][8] through conifold transitions [17]. In fact it has been conjectured that *all* Calabi-Yau compactifications may be connected in this way. In

order to prove a similar statement for moduli spaces of N=2 heterotic compactifications, the classes of theories that one has to connect are even more disparate.

The simplest Calabi-Yau compactification which yields an N=2 heterotic compactification is  $K_3 \times T^2$ . Different choices of gauge bundles yield theories with different spectra, but we have seen that often by moving to a special point in the vectors' moduli space where charged hypermultiplets become massless, we can move to a new partial Higgs phase and obtain a model with different numbers of vectors and hypermultiplets. Unlike the situation in [8], these hypermultiplets arise in the perturbative spectrum of the heterotic string.

However, beyond the  $K_3 \times T^2$  compactifications, there are many asymmetric orbifold compactifications which yield N=2 heterotic theories in four dimensions. These are naively completely non-geometrical (left and right movers live on different spaces!) and one might despair of obtaining them as smooth deformations of  $K_3 \times T^2$  compactifications.

For example, one can easily write down, among many other possibilities, orbifolds with (0, 24) and (4, 20) which one cannot obtain in the manner discussed thus far by choosing stable bundles over  $K_3$ . Consider for example a compactification on a Narain lattice given by  $\Gamma^{4,20} \oplus \Gamma^{2,2}$  where  $\Gamma^{p,p+8k}$  are arbitrary self-dual even lattices. If we choose  $\Gamma^{4,20}$  to correspond to  $SO(8) \times SO(40)$  weight lattices (with difference in the root lattice [26]) and just consider the  $Z_2$  reflection to act only on the  $SO(8)$  part together with a  $v^2 = 0$  shift in  $\Gamma^{2,2}$  we get a model with no hypermultiplets and with  $22 + 2 = 24$  vector fields. The (4, 20) model can be obtained by considering  $\Gamma^{4,4} \oplus \Gamma^{2,18}$  and modding out by a reflection in  $\Gamma^{4,4}$  accompanied by a shift in  $\Gamma^{2,18}$  (to make the left-right level matching work). This gives us 4 hypermultiplets from the moduli of  $\Gamma^{4,4}$  and  $2 + 18 = 20$   $U(1)$  gauge fields. This latter model does correspond to a model that can be realized geometrically as the holonomy action is left-right symmetric.

The model with (4, 20) can also be obtained in another way once we recall one of the most characteristic features of the heterotic string – special singular points in moduli space where enhanced “stringy” gauge symmetries arise! Consider for example the  $K_3$  orbifold obtained as  $T^4/Z_2$ . The spectrum of this theory has been worked out in detail in [27]. At this orbifold point an extra  $SU(2)$  gauge symmetry appears. Using the standard embedding one finds hypermultiplets with the following  $E_7 \times SU(2)$  charges:

$$8 \text{ (}\mathbf{56}, \mathbf{1}\text{)}, \quad 1 \text{ (}\mathbf{56}, \mathbf{2}\text{)}, \quad 32 \text{ (}\mathbf{1}, \mathbf{2}\text{)}, \quad 4 \text{ (}\mathbf{1}, \mathbf{1}\text{)} . \quad (3.2)$$

These include the familiar ten **56**s and 65 moduli of the standard embedding, but some of them are paired in  $SU(2)$  doublets.<sup>2</sup> There are also three additional Higgses for the  $SU(2)$  gauge symmetry. If we break the  $SU(2)$  by Higgsing, we recover the  $(65, 19)$  theory discussed above. However, we can also give the scalar in the  $U(1) \subset SU(2)$  vector multiplet a VEV, moving to the Coulomb phase of the enhanced gauge symmetry. This will give all of the  $SU(2)$  doublets masses, leaving us with 20 vectors (from the Cartan piece of  $E_8 \times E_7 \times SU(2) \times U(1)^4$ ) and 4 hypermultiplets! This reproduces the  $(4, 20)$  of the orbifold above.

Similarly, the  $K_3$  moduli space at certain Gepner points develops an extra rank 5 enhanced gauge symmetry [29]. For example the 1<sup>6</sup> Gepner model has an extra  $U(1)^5$  gauge symmetry<sup>3</sup>. It is easy to check that all of the hypermultiplets are charged under one or more of these  $U(1)$ s, so moving to a generic point in the moduli space of the vectors in this theory yields a model with  $(0, 24)$ , just like the asymmetric orbifold example above. Such examples make it natural to conjecture that most or all  $N=2$  heterotic compactifications are connected by such transitions from Higgs to Coulomb phases, sometimes going through points with enhanced stringy gauge symmetries.

## 4. Heterotic/Type II Duality

### 4.1. General Remarks

As previously mentioned, some of the examples discussed so far do have potential Calabi-Yau “duals,” but these manifolds are much too complicated to allow for a really convincing check of any duality conjecture. On the other hand, we have seen that by Higgsing using charged fields we can reduce the rank of the unbroken gauge group. If we reduce the rank sufficiently, we obtain a model which would be dual to a type IIA compactification on a Calabi-Yau with  $b_{11}$  small. Searching for such Calabi-Yaus is easier because they are relatively rare. Moreover for such examples, the exact structure of the moduli space of  $(1,1)$  forms has been determined using mirror symmetry<sup>4</sup>, and we could

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<sup>2</sup> The  $SU(2)$  doublets pair some  $K_3$  moduli with moduli of the vacuum gauge bundle, showing that there is a duality analogous to that of [28] for these  $K_3$  theories.

<sup>3</sup> The same can be done using the  $T^4/Z_2$  example above by choosing the circles to correspond to  $SU(2)$  symmetry points.

<sup>4</sup> Alternatively, one could study type IIB strings on the mirror, in which case the tree level sigma model computes the exact structure of moduli space.

therefore give stringent tests of any duality conjecture between the string tree level moduli space of  $(1,1)$  forms on such a Calabi-Yau and the exact quantum moduli space of vector multiplets in a given heterotic model.

We have already seen a hint of heterotic/type II duality in §3. There we saw special points in the moduli space of vectors where charged hypermultiplets become massless; giving them VEVs Higgses the gauge symmetry and moves us on to a new branch of the moduli space. In the spirit of [2], one would expect the “quantum” version of this story to change: one would expect a “magnetic Higgs phase” in which charged solitons condense to be responsible for the new branch of moduli space. And the charged black holes of [7][8] are amenable to exactly such an interpretation. With these general remarks out of the way, let us move on to construct some explicit examples of heterotic theories which are dual to type II theories on Calabi-Yau threefolds.

In any  $N=2$  heterotic string, there will be at least two vectors (one vector multiplet) – the graviphoton and the vector in the supermultiplet of the dilaton. If we want, classically, to have points with nonabelian gauge symmetry, then we need at least one more vector, making it desirable to study heterotic models with 3 vectors. A type IIA string compactified on a Calabi-Yau with  $b_{11} = 2$  would also give rise to 3 vectors (including the graviphoton). Similarly, a heterotic model in which all of the gauge symmetry came from the  $U(1)^4$  of the torus (which is enhanced to nonabelian groups at special points) would be dual to a Calabi-Yau with  $b_{11} = 3$ , perhaps. This makes the strategy clear – we should look for heterotic models with 3 or 4 vectors, and try to match them to type II compactifications on known Calabi-Yau manifolds. After discussing in detail specific examples for both the rank three and four cases, in §4.5 we list several more examples of heterotic compactifications with the right spectra to be dual to type II compactifications on known Calabi-Yau threefolds. We have corrected §4.5 in light of comments and work which appeared after the preprint version of this paper. Therefore, we only give examples of heterotic compactifications with spectra which match those of type II compactifications on Calabi-Yau threefolds which are  $K3$  fibrations. Such threefolds appear to be the relevant class in understanding heterotic/type II duality [30][31], and as the list of such manifolds in [30] is quite short, any matches are highly suggestive.

#### 4.2. A Rank Three Example and Its Dual

We begin our search for heterotic/type II dual pairs with a rank three example. Since the most familiar  $K_3 \times T^2$  compactifications automatically yield at least a rank four gauge group (from the  $U(1)^4$  of the torus), we must somehow remove some of the gauge symmetries coming from the torus to get only a rank three gauge group at low energies.

One way of doing this is to start with the  $E_8 \times E_8$  string and *first* compactifying to eight dimensions on a 2-torus with  $\tau = \rho$ , which yields an  $E_8 \times E_8 \times SU(2) \times U(1)^3$  gauge group. Upon further compactification on  $K3$  down to four dimensions, we can now use the extra  $SU(2)$  gauge symmetry in satisfying (2.4) by also turning on gauge fields of this  $SU(2)$ . We embed  $SU(2)$  bundles with  $\int_{K_3} c_2 = 10$  in each  $E_8$  and an  $SU(2)$  bundle with  $\int_{K_3} c_2 = 4$  in the  $SU(2)$ . This leaves an  $E_7 \times E_7 \times U(1)^3$  gauge symmetry. The hypermultiplets include 3 **56**s of each  $E_7$  and 59 gauge neutral moduli. Higgsing the  $E_7$ s completely yields  $2 \times (3 \times 56 - 133) = 70$  extra gauge neutral moduli, leaving a spectrum of 129 hypermultiplets and 3 vectors (2 vector multiplets). It may appear to the reader that we have a lot of room in choosing the  $c_2$ 's of various  $SU(2)$ 's. This is not so. In fact, if we wish to break the full  $E_8 \times E_8 \times SU(2)$  by Higgsing there is only one other choice! This rigidity is in accord with the relative scarcity of low  $b_{11}$  Calabi-Yau manifolds.

We obtained this model by compactifying to eight dimensions on a torus with  $\tau = \rho$ , and then breaking the resulting  $SU(2)$  enhanced gauge symmetry completely. This removes the modulus which would take one away from  $\tau = \rho$ , leaving one with a moduli space for these  $\tau = \rho$  tori which consists of only one copy of the fundamental domain of  $SL(2, \mathbb{Z})$ . When  $\tau = \rho = i$  there is an  $SU(2) \times SU(2)$  enhanced gauge symmetry on the torus, and one of these enhanced  $SU(2)$ s will still be present in our theory. Therefore, if we denote the two vector multiplets in this compactification by  $\tau$  and  $S$  ( $S$  denotes the dilaton; the third  $U(1)$  comes from the graviphoton), then we see that  $\tau = i$  should be a point where one obtains pure  $SU(2)$   $N=2$  gauge theory. Thus, we are studying the closest heterotic string analogue of the  $N = 2$  Yang-Mills theory recently solved in the work of Seiberg and Witten [1].

Can we find a conjectural type II dual for this heterotic theory? There is apparently a unique known Calabi-Yau manifold  $M$  with  $b_{11} = 2$  and  $b_{21} = 128$ . It is the hypersurface of degree 12 in  $WP_{1,1,2,2,6}^4$  (another realization of this manifold is discussed in [32]). Given the scarcity of known Calabi-Yau manifolds with  $b_{11} = 2$ , this match is highly suggestive. Luckily, the moduli space of vector multiplets in type IIA compactification on this manifold

has been studied in great depth in [33][34] using mirror symmetry. We will now provide extremely strong evidence that the heterotic compactification described above is dual to the type IIA string on  $M$ . In fact, we claim the moduli space of vector multiplets on  $M$  is the quantum moduli space of the dual heterotic string!

As we remarked above, in the classical heterotic theory the moduli space of the  $\tau$  vector multiplet is given by one copy of the fundamental domain of  $SL(2, \mathbb{Z})$ . In the full theory with  $\tau$  and  $S$  we therefore expect that as  $S \rightarrow \infty$  (weak coupling) we should find a copy of the fundamental domain of  $SL(2, \mathbb{Z})$  embedded in the moduli space, even the exact quantum moduli space. We also expect a singular point on this moduli space, when  $\tau = i$ , where classically the  $U(1)$  gauge symmetry is enhanced to  $SU(2)$ .

It is useful at this point to review some results of [33][34]. The complex moduli of the mirror of  $M$  can be represented roughly speaking as  $\phi$  and  $\psi$  in the defining polynomial

$$p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^6 z_2^6 \quad (4.1)$$

where the  $z_i$  are the coordinates of the  $WP^4$  (our choice of notation in this equation for the complex moduli of the mirror follows [34]). The authors of [34] actually find it convenient to introduce the large complex structure coordinates

$$Y_1 = \frac{(12\psi)^6}{2\phi} \quad (4.2)$$

$$Y_2 = (2\phi)^2. \quad (4.3)$$

They then compute  $Y_{1,2}$  as functions of the (exponentiated) complexified Kahler forms  $q_{1,2}$  of  $M$ .

We will also find it convenient to use results of [33], so we need to provide a notational dictionary. In appendix A.1 of [33], the notation  $\bar{x}$  and  $\bar{y}$  is used for the complex moduli of the mirror. The dictionary translating between [33] and [34] is simply given by

$$\bar{x} = -\frac{1}{864} \frac{\phi}{\psi^6} = -\frac{1728}{Y_1} \quad (4.4)$$

$$\bar{y} = \frac{1}{\phi^2} = \frac{4}{Y_2}. \quad (4.5)$$

We must now decide what we should check, to decide whether or not this type II moduli space is providing a description of the quantum moduli space of our heterotic model. Classically, the heterotic theory has an  $SU(2)$  enhanced gauge symmetry at  $\tau = i$ .

Therefore we should expect to recover a structure reminiscent of  $SU(2)$  N=2 gauge theory in the weak coupling  $S \rightarrow \infty$  limit where the gravitational effects are not significant. More precisely, it is shown in [1] that in the case of  $SU(2)$  N=2 gauge theory, the isolated singular point in the perturbative theory (where  $SU(2)$  is restored) splits into two singular points in the full quantum theory, where monopoles become massless. So we also expect that as the string coupling is turned on, we will see a single singular point at  $\tau = i$  being split into two singular points.

Does this picture hold? The discriminant locus of this model has been studied by [33][34] where they find it is given by

$$(1 - \bar{x})^2 - \bar{x}^2 \bar{y} = 0 . \quad (4.6)$$

The first thing we notice is that (4.6) is quadratic in  $\bar{x}$ , which is necessary for a picture like that of [1] to hold. This suggests that  $\bar{x}$  parametrizes the  $\tau$  space in some way. At  $\bar{y} = 0$  the two solutions for  $\bar{x}$  merge, so we should identify  $\bar{y} = 0$  with  $S = \infty$  and  $\bar{x} = 1$  with the  $SU(2)$  point. This means that  $\bar{x} = 1$  should be identified with  $\tau = i$ , at least at  $\bar{y} = 0$ . Note that  $\bar{y} = 0$  corresponds to the large radius limit of  $M$ , where the leading behavior of the metric on moduli space is the same as that expected for the dilaton.

These tests have given us some idea of what to expect, in terms of identifying the type II parametrization of the moduli space with the heterotic parametrization. Now we give much stronger evidence for the interpretation offered above, based on a surprising observation of [34]. There, in §7, it was noted that at  $\bar{y} = 0$  the mirror map giving the relation between  $\bar{x}$  and the special coordinates on the moduli space of  $M$  is given by the elliptic  $j$ -function! More precisely

$$\bar{x} = \frac{1728}{j(\tau_1)} \quad (4.7)$$

where  $\tau_1$  parametrizes a kahler modulus of  $M$  in this limit. This was noted as an unexplained “curiosity” by the authors of [34]. From (4.7) we see that, since  $j(i) = 1728$ , the point corresponding to  $\bar{x} = 1$  is precisely  $\tau_1 = i$ ! This means we should identify  $\tau = \tau_1$  at least for weak coupling. Then (4.7) implies that

$$\bar{x} = \frac{1728}{j(\tau)} . \quad (4.8)$$

Since  $\bar{y} = 0$  corresponds to  $S = \infty$ , we also propose

$$\bar{y} = e^{-S} \quad (4.9)$$

to leading order in the coupling.

But we can check more. Since we are conjecturing that the type II vector moduli space is the fully quantum corrected version of the heterotic vector moduli space, we should also check that the one-loop corrections to the prepotential are reflected in the  $\bar{y} \rightarrow 0$  limit of the moduli space of  $M$ . The one-loop correction to the prepotential  $F$  in similar N=2 heterotic theories has been computed in [35][36] (and one could directly compare with their computations in the next example). For our example, predictions for the full non-perturbative third derivatives of the prepotential directly follow from the formulas of [33][34]. Specializing to the  $\bar{y} \rightarrow 0$  limit, we should be able to recover the string one-loop corrections. In particular, using the formulas of appendix A.1 of [33] (and remembering to restore the factor of the discriminant (4.6) in the denominators!) and the dictionary (4.8)(4.9), we see we are predicting

$$F_{\tau\tau\tau} = \frac{j_\tau^3}{j(\tau)(j(\tau) - j(i))^2} \quad (4.10)$$

$$F_{S\tau\tau} = \frac{j_\tau^2}{j(\tau)(j(\tau) - j(i))} \quad (4.11)$$

where  $j_\tau$  denotes the derivative of  $j$  with respect to  $\tau$ .  $F_{SS\tau}$  and  $F_{SSS}$  vanish in the limit of weak coupling that we are considering, as expected from perturbative string theory.

Actually, the formulas (4.10) and (4.11) are not quite the final story. In string theory, we expect the gauge coupling function, obtained by taking two derivatives of the prepotential, to really be a *function* (and not a section of some bundle). This means that the third derivatives  $\tilde{F}_{\tau\tau\tau}$  and  $\tilde{F}_{S\tau\tau}$  for our example should really be modular forms of weight 2 and 0 with respect to the  $SL(2, \mathbb{Z})$  symmetry acting on  $\tau$ . This has been obscured by the gauge choice made for the ‘Yukawa couplings’, but we can restore the correct modular properties by simply recognizing that the appropriate gauge for the ‘Yukawa couplings’ must be<sup>5</sup>

$$\tilde{F}_{\tau\tau\tau} = \frac{F_{\tau\tau\tau}}{E_4(\tau)} \quad (4.12)$$

$$\tilde{F}_{S\tau\tau} = \frac{F_{S\tau\tau}}{E_4(\tau)} \quad (4.13)$$

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<sup>5</sup> It would be desirable to see why this is natural also from the type II side using the results of [33][34]. Note that the gauge independent quantity  $F_{\tau\tau\tau}/F_{S\tau\tau} = j_\tau/(j(\tau) - j(i))$  agrees with what one expects from the heterotic string.

where  $E_4(\tau)$  is the fourth Eisenstein series (which is 1/240-th of the theta function for the  $E_8$  lattice), and we know it must appear due to the uniqueness of weight four modular forms. So the true formulas we are predicting for the third derivatives of the prepotential are:

$$\tilde{F}_{\tau\tau\tau} = \frac{j_\tau^3}{E_4(\tau)j(\tau)(j(\tau) - j(i))^2} \quad (4.14)$$

$$\tilde{F}_{S\tau\tau} = \frac{j_\tau^2}{E_4(\tau)j(\tau)(j(\tau) - j(i))} . \quad (4.15)$$

How do (4.14) and (4.15) compare with expectations? Based on its singularity properties and asymptotic behavior alone (for similar arguments see e.g. [37][35][36]), we know we can fix the gauge coupling function  $\tilde{F}_{\tau\tau}$  to be

$$\tilde{F}_{\tau\tau} \sim S + \log(j(\tau) - j(i)) . \quad (4.16)$$

This means that we expect

$$\tilde{F}_{\tau\tau\tau} \sim \frac{j_\tau}{(j(\tau) - j(i))} . \quad (4.17)$$

Remarkably, the elliptic  $j$ -function satisfies the identity

$$j_\tau^2 = -960\pi^2 j(\tau)(j(\tau) - j(i))E_4(\tau) \quad (4.18)$$

which makes it clear that (4.14) is in fact of the expected form (4.17)! Similarly, we see using the identity (4.18) that  $F_{S\tau\tau} \sim 1$  at weak coupling, as expected. So we see our results agree not only with the classical heterotic string picture but also with the expected one-loop string correction.

In identifying the algebraic variables  $\bar{x}$  and  $\bar{y}$  with  $\tau$  and  $S$  beyond the leading order it is natural to conjecture that they continue to be related by the mirror map.

### 4.3. New Stringy Phenomena

So far we have given strong evidence for the identification of our heterotic model with the Type IIA string on  $M$ . Our main interest is to use these results to see how nonperturbative string corrections modify the classical string picture. For example, there might be qualitatively new effects due to the presence of gravity. Here we will make a few preliminary remarks, leaving the full story to future work.

For finite  $\bar{y}$ , the singular locus already “knows” that the moduli space is the fundamental domain of  $SL(2, Z)$ . But for very small  $\bar{y}$  this may not be the case. In this region of

infinitesimal  $\bar{y}$ , we can hope to recover in much greater detail the results of [1] (perhaps by considering a double scaling limit). We might also see stringy corrections to these results. This is presently under investigation [38].

Alternatively, for finite  $\bar{y}$  we might also probe qualitatively new stringy modifications due to nonperturbative effects present in string theory but absent in field theory. In fact, in addition to the discriminant locus (4.6) which is the locus of conifold singularities, there is an additional singular locus at  $\bar{y} = 1$  where the manifold acquires a complicated point singularity [34].<sup>6</sup> Note that with the identification (4.9) the locus  $\bar{y} = 1$  corresponds to  $S \rightarrow 0$ , i.e. infinitely strong coupling. Moreover, along this locus the threefold becomes birationally equivalent to the  $(2, 6)$  complete intersection in  $WP_{1,1,1,1,3}^5$  [34], suggesting that there should be a smooth transition similar to [8]. It would be interesting to unravel the physics of this transition, from the type II side. One concrete hint is the singularity structure in the one-loop computation of  $R^2$ , which gives the net number of massless hypermultiplets [9].

An interesting point to notice is that there is an extra  $Z_6$  symmetry for all  $\bar{y}$  at  $\bar{x} = \infty$ , which at  $\bar{y} = 0$  corresponds to  $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ . This is the quantum  $Z_6$  symmetry of the Landau-Ginzburg theory [34], and we see that it survives string nonperturbative effects. Moreover, if we tune the coupling constant properly (and with a particular choice of  $\tau$  for the heterotic compactification) we get an enhanced  $Z_{12}$  symmetry point on moduli space (corresponding to  $\phi = \psi = 0$ ).

#### 4.4. A Rank Four Example and its Dual

Having met with success in finding a dual for a rank three example, we now move on to a rank four example. Let us start with the heterotic  $E_8 \times E_8$  string. Embed a rank 2 bundle with  $\int_{K_3} c_2(V) = 12$  in each  $E_8$ ; this gives rise to a theory with 4 **56**s in each  $E_7$  and a total of 62 gauge neutral moduli hypermultiplets. Now Higgs both  $E_7$ s completely by giving VEVs to the charged fields. This gives an extra  $4 \times 56 - 133 = 91$  neutral fields from each factor, leaving us with 244 hypermultiplets and just the  $U(1)^4$  gauge symmetry of the torus (a  $U(1)^2$  of which is enhanced to  $SU(2) \times U(1)$ ,  $SU(2) \times SU(2)$  and  $SU(3)$  at special points).

We would like to find a Calabi-Yau manifold  $X$  with  $b_{11} = 3$  and  $b_{21} = 243$ , on which type II strings could be dual to the heterotic theories we have just described. And in fact,

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<sup>6</sup> We would like to thank S. Hosono for helpful correspondence on this point.

such a manifold  $X$  does exist! It is the degree 24 hypersurface in  $WP_{1,1,2,8,12}^4$  which has been studied (using mirror symmetry) in [33].

$X$  is defined by an equation of the form

$$a_1 z_1^2 + a_2 z_2^3 + a_3 z_3^{12} + a_4 z_4^{24} + a_5 z_5^{24} - 12\alpha z_1 z_2 z_3 z_4 z_5 - 2\beta z_3^6 z_4^6 z_5^6 - \gamma z_4^{12} z_5^{12} \quad (4.19)$$

where the  $z_i$  are the weighted projective space coordinates. It is actually convenient to define

$$\bar{x} = -\frac{1}{3456} \frac{a_1^3 a_2^2 \beta}{\alpha^6} \quad (4.20)$$

$$\bar{y} = \frac{4a_4 a_5}{\gamma^2} \quad (4.21)$$

$$\bar{z} = -\frac{1}{2} \frac{a_3 \gamma}{\beta^2} \quad (4.22)$$

which serve as the complex structure coordinates of the mirror.

There are several things we expect to be true for this model, which we would like to check. For example, we expect that in an appropriate weak coupling limit, there should be a copy of the  $\tau$  and  $\rho$  moduli spaces of  $T^2$  (a product of two copies of the fundamental domain of  $SL(2, Z)$ ) embedded in the Kahler moduli space of  $X$ . In this weak coupling limit, we can also make several statements about the singularity structure. At generic points in the moduli space, the left-movers of the toroidal compactification are responsible for a  $U(1)^2$  gauge symmetry. But on the locus  $\tau = \rho$  this is enhanced to an  $SU(2) \times U(1)$  gauge symmetry while the points  $\tau = \rho = i$  and  $\tau = \rho = \frac{1}{2} + i\frac{\sqrt{3}}{2}$  have further enhancement to  $SU(2) \times SU(2)$  and  $SU(3)$  gauge symmetry, respectively. We therefore expect a singular locus, at very weak coupling, which looks like the  $\tau = \rho$  copy of the fundamental domain of  $SL(2, Z)$ , with two special points.

We now give a description of the good coordinates on the moduli space. The  $SL(2, Z)$  invariances tell us that we should work with  $j(\tau)$  and  $j(\rho)$ , as they map the fundamental domain of  $SL(2, Z)$  bijectively to the complex plane. There is also a  $Z_2$  exchanging  $\tau$  and  $\rho$ , so the really good coordinates are

$$u = j(\rho) + j(\tau) \quad (4.23)$$

$$v = 4j(\rho) \times j(\tau) . \quad (4.24)$$

The codimension one singularity, at weak coupling, should be at  $\tau = \rho$  which is the same as the locus

$$u^2 - v = 0 . \quad (4.25)$$

We expect to have a double point at weak coupling when  $\tau = \rho$  for *generic*  $\tau$  and  $\rho$ , since we have an  $SU(2)$  enhanced gauge symmetry. So we really expect a discriminant locus of the form

$$(u^2 - v)^2 = 0 \tag{4.26}$$

at zero coupling.

The discriminant locus of  $X$  is given by <sup>7</sup>

$$(1 - \bar{x})^4 - 2\bar{x}^2\bar{z}(1 - \bar{x})^2 + \bar{x}^4\bar{z}^2(1 - \bar{y}) = 0 . \tag{4.27}$$

First of all, note that at  $\bar{y} = 0$  the equation becomes

$$((1 - \bar{x})^2 - \bar{x}^2\bar{z})^2 = 0 \tag{4.28}$$

which is of the form (4.26) expected from the heterotic string, where roughly speaking  $u \sim \bar{x}$  and  $v \sim \bar{z}$ . To check in more detail this duality, one would like to find a relationship at  $\bar{y} = 0$  between  $\bar{x}$ ,  $\bar{z}$ , and the  $j$  functions. At our request, Hosono has investigated this question using the results of [33] and has found some preliminary evidence suggesting such a relationship. Very detailed checks for this example must await further investigation of this question [38]. Nevertheless, we will try to proceed with some further qualitative checks.

Another qualitative fact we would like to explain is the presence of codimension two singularities which should occur (at weak coupling) at the  $SU(2) \times SU(2)$  and  $SU(3)$  points. In fact, there are further singular loci in the moduli space of  $X$ , one of which is given by

$$(1 - \bar{z})^2 - \bar{y}\bar{z}^2 = 0 . \tag{4.29}$$

This intersects weak coupling ( $\bar{y} = 0$ ) at  $\bar{z} = 1$ . Furthermore, there is an intersection of this locus with the  $\tau = \rho$  locus at two points. One intersection occurs at  $\bar{x} = \infty, \bar{z} = 1$  where there is a  $Z_6$  symmetry; we would like to identify this with the  $SU(3)$  point. The other intersection is at  $\bar{x} = \frac{1}{2}, \bar{z} = 1$ . We would like to identify this point with the  $SU(2) \times SU(2)$  point. The symmetry properties of this moduli space unfortunately have not been worked out. This would be crucial for a better understanding of the nature of the  $\bar{z} = 1$  singularity. However, for generic  $\tau, \rho$  with  $\epsilon = \tau - \rho$  small, we have checked using the results of [33] that

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<sup>7</sup> There is a misprint in the discriminant locus presented in appendix A.1 of the hep-th version of [33], corrected in the published version. We thank S. Hosono for informing us of this.

the requisite singularity structure of the third derivatives of the prepotential is reproduced. In particular,

$$F_{\epsilon\epsilon\epsilon} \sim \frac{1}{\epsilon} \quad (4.30)$$

$$F_{\epsilon\epsilon S} \sim 1 \quad (4.31)$$

with  $F_{\epsilon SS}$  and  $F_{SSS}$  vanishing.

#### 4.5. More Potential Dual Pairs

Since we have been studying very special examples, the reader might wonder how easy it is to find other examples where one can propose a (testable) duality between heterotic and type II compactifications. As mentioned in §4.1, this subsection has been revised since the preprint version of this paper. In light of the very recent work suggesting that CY threefolds which are K3 fibrations play an important role in heterotic/type II duality, it makes sense to try and find heterotic duals for the K3 fibrations listed in [30] (which is a highly restricted list, compared to the lists of all known CY spaces). At the level of simply matching the numbers of hypermultiplets and vector multiplets, it is not difficult to construct many such examples, and we provide some below (all on the list in [30]). It would be extremely interesting to explore some of these examples in more detail to see if one's expectations for the heterotic models are reflected in the type II moduli spaces, as in §4.2. Of course our construction of examples is by no means exhaustive. It might prove beneficial to check other sorts of examples, or to find a general recipe linking the heterotic constructions to the dual manifolds.

One nice class of heterotic N=2 compactifications can be constructed as follows. Start with a compactification to nine dimensions on a Narain lattice  $\Gamma^{1,17}$  which gives  $SO(34) \times U(1)$  gauge group. Further compactify to eight dimensions on a circle, including Wilson lines which break the  $SO(34)$  to an  $SO(34-2n) \times SO(2n)$  in a way consistent with level-matching [39]. This yields a theory in eight dimensions with  $SO(34-2n) \times SO(2n) \times U(1)^3$  gauge group, and upon compactification to four dimensions on  $K_3$  we can compute the resulting spectrum using the techniques of §2. We call the factors of the vacuum gauge bundle embedded in  $SO(34-2n)$  and  $SO(2n)$   $V_1$  and  $V_2$  (where we keep the convention  $n \leq 8$ ), and we use the notation

$$d_1 = \int_{K_3} c_2(V_1), \quad d_2 = \int_{K_3} c_2(V_2) . \quad (4.32)$$

Some examples which fall into this category are the following:

- 1) Consider the  $n = 6$  case. In the further compactification on  $K_3$ , choose  $V_{1,2}$  to be  $SU(2)$  bundles with  $d_{1,2} = 16$  and  $8$ . After Higgsing maximally, leaving an  $SO(6) \times SO(4) \times U(1)^3$  gauge group unbroken, the spectrum at generic points in the moduli space is  $(144, 8)$ . There is a known K3 fibration with  $b_{11} = 7, b_{21} = 143$  – the hypersurface in  $WP_{1,1,4,4,10}^4$ .
- 2) Consider the  $n = 6$  case, and in the further compactification on  $K_3$  choose  $V_1$  to be a  $d_1 = 20$   $SU(2)$  bundle and  $V_2$  to be a  $d_2 = 4$   $SU(2)$  bundle. After Higgsing as much as possible one is left with a generic spectrum of  $(195, 9)$ . The Calabi-Yau hypersurface in  $WP_{1,1,4,8,14}^4$  is a K3 fibration with  $b_{11} = 8, b_{21} = 194$ .
- 3) Consider the construction above, with the compactification to 8 dimensions done on a circle at the self-dual radius with no Wilson lines. Then the gauge group is  $SO(34) \times SU(2) \times U(1)^2$ . Now in compactifying on K3 to four dimensions, embed rank two bundles with  $\int_{K_3} c_2 = 20$  and  $4$  into the  $SO(34)$  and  $SU(2)$ , respectively. Maximally Higgsing yields a spectrum of  $(195, 9)$  again, so as in example 2) this model is appropriate for a dual to the hypersurface in  $WP_{1,1,4,8,14}^4$ . In particular, it should lie in the same moduli space as example 2).

We can give more examples by following the same strategy we used with  $SO(34)$ , this time with  $SO(32 - 2n) \times SO(2n) \times U(1)^4$ . We adhere to the same notation, and find the following examples which have known candidate Calabi-Yau duals:

- 4) Take  $n = 0$ , compactifying on  $K_3 \times T_2$  in the normal way. Take  $V_1$  to be a rank 2 bundle with  $d_1 = 24$ . One is left with  $SO(28) \times SU(2) \times U(1)^4$  gauge group in four dimensions, with 10 **(28, 2)**s of  $SO(28) \times SU(2)$ . Higgsing as much as possible, one is left with an unbroken  $SO(8) \times U(1)^4$  gauge group and generically a  $(272, 8)$  spectrum. This is appropriate for a dual to the type IIA string compactified on the hypersurface in  $WP_{1,1,4,12,18}^4$ . This example (and the following one) was mentioned in [31], where an error in the preprint version of this paper was corrected.
- 5) Take  $n = 0$ , that is consider the  $SO(32)$  string compactified to eight dimensions on a torus, but take the torus to sit at  $\tau = \rho = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ . Then the gauge symmetry is  $SO(32) \times SU(3) \times U(1)^2$ . Embed an  $SU(2)$  bundle with  $\int_{K_3} c_2 = 18$  in the  $SO(32)$  and an  $SU(3)$  bundle with  $\int_{K_3} c_2 = 6$  in the  $SU(3)$ . After Higgsing, one obtains the spectrum  $(165, 9)$  which would arise from type IIA compactification on a manifold with  $b_{11} = 8, b_{21} = 164$ . Such a K3 fibration does exist – the hypersurface in  $WP_{1,1,4,6,12}^4$ .

We can also use the  $E_8 \times E_8$  string as well as the more general groups one can get in Narain compactification to eight dimensions to try and generate more examples. Here we present some simple examples using these other possibilities. Below,  $V_{1,2}$  and  $d_{1,2}$  refer to the bundles embedded in the two  $E_8$ s.

6) Start with the “standard embedding” compactification on  $K3 \times T^2$ , i.e. with  $V_1$  an  $SU(2)$  bundle and  $d_1 = 24$ . Now Higgs the unbroken  $E_7$  completely – the resulting spectrum  $(492, 12)$  would arise from type IIA strings on a Calabi-Yau with  $b_{11} = 11, b_{21} = 491$  and there is such a space, the hypersurface in  $WP_{1,1,12,28,42}^4$  (note that  $b_{21}$  of this space is given incorrectly in the list of [30], although  $b_{11}$  and the Euler character are given correctly there).

7) Start with  $E_8 \times E_8 \times SU(2) \times U(1)^3$  in eight dimensions by compactifying on a torus with  $\tau = \rho$ . In the further reduction on K3 choose  $V_1$  a rank two bundle with  $d_1 = 20$  and also embed a rank two bundle with  $\int_{K3} c_2 = 4$  into the  $SU(2)$ . After Higgsing here, one obtains a  $(377, 11)$  spectrum appropriate to a potential dual for the hypersurface in  $WP_{1,1,8,20,30}^4$ .

8) Start with  $E_8 \times E_8 \times SU(3) \times U(1)^2$  by compactifying to eight dimensions on a torus with  $\tau = \rho = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ . Choose  $V_1$  and  $SU(2)$  bundle with  $d_1 = 18$  and also embed rank three bundle with  $\int_{K3} c_2 = 6$  in the  $SU(3)$  when you further compactify on K3. After Higgsing this is a potential dual to the  $b_{11} = 9, b_{21} = 321$  manifold on the list in [30], the hypersurface in  $WP_{1,1,6,16,24}^4$ .

9) Start in eight dimensions just as in example 8), but this time choose  $V_{1,2}$  both of rank two with  $d_1 = 10, d_2 = 8$ , and also embed an  $SU(3)$  bundle with  $\int_{K3} c_2 = 6$  into the  $SU(3)$ . Completely Higgs the first  $E_8$  and Higgs the other down to  $SO(10)$ , then go to a generic point in the moduli space of the vectors. This yields a potential dual to the complete intersection of degree 8 and 12 hypersurfaces in  $WP_{1,1,2,4,6,6}^5$ , which has  $b_{11} = 6, b_{21} = 98$ .

10) Start with the Narain compactification yielding  $E_8 \times SO(20) \times U(1)^2$  gauge group in eight dimensions. Embed rank two bundles with  $\int_{K3} c_2 = 10$  and 14 into the  $E_8$  and the  $SO(20)$  factors in the further compactification on K3. Higgsing as much as possible (leaving an  $SO(6)$  subgroup of the  $SO(20)$  unbroken), one finds a generic spectrum  $(149, 5)$ , which could describe a heterotic dual to type IIA strings on the hypersurface in  $WP_{1,1,2,4,8}^4$ .

11) Start with the compactification on a special torus yielding  $E_8 \times E_8 \times SU(3) \times U(1)^2$  gauge group in eight dimensions. Choose  $V_{1,2}$  to be rank two bundles with  $d_1 = 10$  and

$d_2 = 8$ , and embed a rank three bundle with  $\int_{K3} c_2 = 6$  into the  $SU(3)$ . After Higgsing the first  $E_7$  completely and the second  $E_7$  down to  $SO(8)$ , move to a generic point in the moduli of the vectors. This leaves a spectrum of  $(102, 6)$ , appropriate for the dual of the type IIA string compactified on the Calabi-Yau hypersurface in  $WP_{1,1,2,4,4}^4$  which has  $b_{11} = 5, b_{21} = 101$ .

Looking through the list of threefold K3 fibrations for which we have found potential heterotic duals, one is struck by certain phenomenological patterns which the weights of the relevant weighted projective spaces exhibit. For example, starting from some examples with low  $b_{11}$  and shifting the weights of the ambient  $WP^4$  by  $(0, 0, 2, 4, 6)$  often yields another example. Hopefully this is an indication that a general recipe connecting certain classes of Calabi-Yaus to their heterotic duals is within our reach in the near future.

We would like to close by discussing one other type II model which we believe may be describing the nonperturbative structure of  $SU(2)$  N=2 gauge theory with  $N_f = 1$  matter hypermultiplet in the **2** of  $SU(2)$ . In such a theory, we expect that the classical moduli space of the vectors will exhibit a single singular point (where  $SU(2)$  is restored and the charged matter is massless), which splits into *three* singular points in the quantum theory [2]. The Kahler moduli space of the hypersurface in  $WP_{1,1,1,6,9}^4$ , studied in depth in [33][40], is a perfect candidate for the string description of this moduli space. In the coordinates of [33], its discriminant locus is given by

$$(1 - \bar{x})^3 - \bar{x}^3 \bar{y} = 0 . \quad (4.33)$$

Identifying  $\bar{y}$  with the dilaton in the usual way (4.9), we see that this is precisely the singularity structure we expect for N=2  $SU(2)$  gauge theory coupled to a single **2** of  $SU(2)$ . It would be very interesting to find the heterotic dual of the type II compactification on this Calabi-Yau hypersurface.

## 5. Conclusions

We have seen, through examples, that it is possible that the moduli spaces of many different N=2 heterotic vacua are connected in a large web, in a similar way to type II string compactifications [8]. For some of these N=2 heterotic models, we have also been able to construct dual type II string compactifications on Calabi-Yau threefolds.

This raises many interesting questions. The first thing one might wonder about is the generality of the phenomenon – when do we expect a compactification to have dual heterotic

and type II descriptions? Given the bound on the rank of the gauge group attainable in perturbative heterotic strings, it seems unlikely that type II compactifications on Calabi-Yau manifolds with sufficiently large hodge numbers will have heterotic duals. Similarly, we have found examples of heterotic compactifications for which there are no presently known candidate Calabi-Yau duals. In fact, the  $(0, 24)$  example of §3 is an example for which there *cannot* be a dual Calabi-Yau description – that would require  $b_{12} = -1$ ! In other words, such a theory has no dilaton on the type II side. What we envision is that the whole moduli space of  $N=2$ ,  $d=4$  string theories form a connected web. Some regions have type II descriptions, some regions have heterotic descriptions, some regions have both (as we have found), and perhaps some regions have neither. Consider for example the  $E_8 \times E_8$  string broken to  $E_7 \times E_8$  with the standard embedding, as in example 6) of §4.5. At a Gepner point in its moduli space with enhanced  $U(1)^5$  gauge symmetry, we can move (as discussed in §3) off onto a new branch of moduli space which gives rise to the spectrum  $(0, 24)$ . On the other hand, we saw that by Higgsing we could get to a theory with spectrum  $(492, 12)$  for which we have proposed a dual type II Calabi-Yau compactification with  $b_{11} = 11, b_{21} = 491$ ! This means that there is a path one can follow from the  $(492, 12)$  type II theory which has a Calabi-Yau description to a region with spectrum  $(0, 24)$  which cannot have a Calabi-Yau description. Similarly, starting from the Calabi-Yau with  $b_{11} = 11, b_{21} = 491$  we can move using conifold transitions [8][17] to a Calabi-Yau with both  $b_{11} + 1$  and  $b_{21} + 1$  larger than the allowed rank 24 of a gauge group in perturbative heterotic strings. We could follow the same path starting from the  $(492, 12)$  heterotic string, reaching a theory without a perturbative heterotic string description!

In this paper, we have focused on using the duality between certain heterotic and type II  $N=2$  compactifications to determine the exact structure of the moduli space of vectors on the heterotic side in terms of the tree level structure on the type II side. Similarly, one could compute the exact structure of the moduli space of hypermultiplets on the type II side using just the tree level results for the moduli space of the heterotic hypermultiplets. Completely understanding the tree level structure on the heterotic side would involve answering certain questions about the moduli spaces of stable holomorphic vector bundles over  $K_3$  and about the moduli spaces of Higgs fields (which are probably related as we can describe the same moduli space in two different ways).

For examples where we have been successful in constructing a dual, there are many things one would like to learn. There is the exciting possibility of probing stringy non-perturbative effects in a very quantitative manner, and perhaps uncovering qualitatively

new physics. Perhaps such effects would have counterparts in other compactifications, for example compactifications with only  $N = 1$  supersymmetry.

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